Modeling Debt
Solving For Debt IRR In Continuous-Time

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In the white paper **Modeling Debt - Solving For Debt IRR In Discrete-Time** we calculated a credit spread in discrete-time. In this white paper we will calculate a credit spread in continuous-time. To that end we will work through the following hypothetical problem from the discrete-time white paper... [2]

Our Hypothetical Problem

We are given the following model parameters...

Table 1: Model Parameters

Symbol	Description	Value
D_0	Debt principal balance at time zero ($)	100,000.00
D_T	Debt principal balance at time T ($)	75,000.00
G	Cost of the guarantee (paid by the borrower) ($)	9,500.00
R	Annual contractual interest rate (%)	6.00
K	Annual risk-adjusted discount rate (%)	6.50
T	Debt term in years (#)	5.00

Our task is to answer the following questions...

Question 1: What is the market value of debt at time zero?

Question 2: What is the IRR on the debt excluding the guarantee?

Question 3: What is the IRR on the debt including the guarantee?

Question 4: What is the credit spread on the debt?

Debt Valuation Equations

We will define the variable κ to be the continuous-time risk-adjusted discount rate. Using the data in Table 1 above, the equation for the discount rate is...

$$\kappa = \ln(1 + K) \tag{1}$$

We will define the variable ϕ to be the continuous-time contractual debt interest rate. Using the data in Table 1 above, the equation for the contractual debt interest rate is...

$$\phi = \ln(1 + R) \tag{2}$$

We will define the variable Δ to be the continuous-time debt service payment factor. Using Equation (2) above and using the data in Table 1 above, the equation for the debt service payment factor is...

$$\text{if...}\ D_T = D_0 \operatorname{Exp}\left\{ -\lambda T \right\} \ \text{...then...}\ \lambda = -\ln\left(\frac{D_T}{D_0}\right)\Big/ T \quad \text{...such that...}\ \Delta = \lambda + \phi \tag{3}$$

Using Equations (2) and (3) above, the equation for debt balance at time $t \le T$ is...

$$D_t = D_0 \operatorname{Exp}\left\{ \left(\phi - \Delta\right) t \right\} \tag{4}$$

We will define the variable $C_{0,T}$ to be cumulative cash flow received by the lender over the time interval $[0,T]$. Using Equations (2), (3) and (4) above, the equation for cumulative cash flow is...

$$C_{0,T} = \int\limits_0^T \Delta\, D_t\, \delta t + D_T = \Delta\, D_0 \int\limits_0^T \mathrm{Exp}\left\{\left(\phi - \Delta\right)t\right\}\delta t + D_T \tag{5}$$

We will define the variable V_0 to be the present value of lender cash flow over the time interval $[0,T]$. Using Equations (1) and (5) above, the equation for the present value of cash flow is...

$$V_0 = \Delta\, D_0 \int\limits_0^T \mathrm{Exp}\left\{\left(\phi - \Delta\right)t\right\}\mathrm{Exp}\left\{-\,\kappa\, t\right\}\delta t + D_T\,\mathrm{Exp}\left\{-\,\kappa\, T\right\} \tag{6}$$

Using Appendix Equation (19) below, the solution to Equation (6) above is...

$$V_0 = \Delta\, D_0\left[1 - \mathrm{Exp}\left\{\left(\phi - \Delta - \kappa\right)T\right\}\right]\left(\kappa + \Delta - \phi\right)^{-1} + D_T\,\mathrm{Exp}\left\{-\,\kappa\, T\right\} \tag{7}$$

We will make the following definitions...

$$A = 1 - \mathrm{Exp}\left\{\left(\phi - \Delta - \kappa\right)T\right\} \text{ ...and... } B = \left(\kappa + \Delta - \phi\right)^{-1} \text{ ...and... } C = \mathrm{Exp}\left\{-\,\kappa\, T\right\} \tag{8}$$

Using the definitions in Equation (8) above, we can rewrite Equation (7) above as...

$$V_0 = \Delta\, D_0\, A\, B + D_T\, C \tag{9}$$

The derivative of Equation (9) with respect to the discount rate κ is...

$$\frac{\delta V_0}{\delta \kappa} = \Delta\, D_0\left(\frac{\delta A}{\delta \kappa}\, B + \frac{\delta B}{\delta \kappa}\, A\right) + D_T\,\frac{\delta C}{\delta \kappa} \tag{10}$$

Using Appendix Equations (20), (21), and (22) below, the solutions to the derivatives in Equation (10) above are...

$$\frac{\delta A}{\delta \kappa} = T\,\mathrm{Exp}\left\{\left(\phi - \Delta - \kappa\right)T\right\} \text{ ...and... } \frac{\delta B}{\delta \kappa} = -\left(\kappa + \Delta - \phi\right)^{-2} \text{ ...and... } \frac{\delta C}{\delta \kappa} = -T\,\mathrm{Exp}\left\{-\,\kappa\, T\right\} \tag{11}$$

To solve for debt IRR we will use the Newton Raphson Method For Solving Non-Linear Equations. [1]

The Answers To Our Hypothetical Problem

Using Equation (1) above and the data in Table 1 above, the equation for the risk-adjusted discount rate is...

$$\kappa = \ln\left(1 + 0.0650\right) = 0.0630 \tag{12}$$

Using Equations (2) above and the data in Table 1 above, the equation for the contractual interest rate is...

$$\phi = \ln\left(1 + 0.0600\right) = 0.0583 \tag{13}$$

Using Equations (3) above and the data in Table 1 above, the equation for the debt service rate is...

$$\lambda = -\ln\left(\frac{75{,}000}{100{,}000}\right)\Big/ 5.00 = 0.0575 \quad \text{...such that... } \Delta = 0.0575 + 0.0583 = 0.1158 \tag{14}$$

Question 1: What is the market value of debt at time zero?

$$V_0 = 0.1148 \times 100{,}000 \times \left[1 - \mathrm{Exp}\left\{\left(0.0583 - 0.1158 - 0.0630\right) \times 5.00\right\}\right]\left(0.0630 + 0.1158 - 0.0583\right)^{-1}$$
$$+\, 75{,}000 \times \mathrm{Exp}\left\{-\, 0.0630 \times 5.00\right\} = 98{,}232.67 \tag{15}$$

Question 2: What is the IRR on the debt excluding the guarantee.

If we set the guess value of κ to Equation (12) above, and F(actual κ) = Debt principal at time zero, then the IRR on the debt via a Newton-Raphson methodology for solving non-linear equations is... [1]

Iteration	new κ	=	κ guess	F(κ)	F(κ guess)	dF(κ guess)
1	0.06293	=	0.05827	98,232.67	100,000.00	-379,564.91
2	0.06297	=	0.06293	98,232.67	98,251.13	-371,663.97
3	0.06297	=	0.06297	98,232.67	98,232.67	-371,580.62
4	0.06297	=	0.06297	98,232.67	98,232.67	-371,580.61

Using the last entry in the table above, the annualized IRR on the debt excluding the guarantee is...

$$\text{IRR excluding guarantee} = \text{Exp}\left\{0.06297\right\} - 1 = 6.50\% \tag{16}$$

Question 3: What is the IRR on the debt including the guarantee.

If we set the guess value of κ to Equation (12) above, and F(actual κ) = Debt principal at time zero minus the value of the guarantee, then the IRR on the debt via a Newton-Raphson methodology for solving non-linear equations is... [1]

Iteration	new κ	=	κ guess	F(κ)	F(κ guess)	dF(κ guess)
1	0.08795	=	0.05827	88,732.67	100,000.00	-379,564.91
2	0.09013	=	0.08795	88,732.67	89,454.30	-332,059.27
3	0.09014	=	0.09013	88,732.67	88,736.17	-328,836.97
4	0.09014	=	0.09014	88,732.67	88,732.67	-328,821.23

Using the last entry in the table above, the annualized IRR on the debt excluding the guarantee is...

$$\text{IRR including guarantee} = \text{Exp}\left\{0.09014\right\} - 1 = 9.43\% \tag{17}$$

Question 4: What is the credit spread on the debt.

Using Equations (16) and (17) above the credit spread (spread to cover credit losses) on the debt is...

$$\text{Credit spread} = 9.43\% - 6.50\% = 2.93\% \tag{18}$$

Appendix

A. The solution to the following integral is...

$$\begin{aligned}
I &= \int\limits_0^T \text{Exp}\left\{\left(\phi - \Delta\right)t\right\} \text{Exp}\left\{-\kappa\, t\right\} \delta t \\
&= \int\limits_0^T \text{Exp}\left\{\left(\phi - \Delta - \kappa\right)t\right\} \delta t \\
&= \frac{1}{\phi - \Delta - \kappa} \text{Exp}\left\{\left(\phi - \Delta - \kappa\right)t\right\} \Bigg[_0^T \\
&= \frac{1}{\phi - \Delta - \kappa}\left[\text{Exp}\left\{\left(\phi - \Delta - \kappa\right)T\right\} - 1\right] \\
&= \left[1 - \text{Exp}\left\{\left(\phi - \Delta - \kappa\right)T\right\}\right]\left(\kappa + \Delta - \phi\right)^{-1}
\end{aligned} \tag{19}$$

B. The solution to the following equation is...

$$A = \frac{\delta}{\delta\kappa} \left[1 - \text{Exp}\left\{ \left(\phi - \Delta - \kappa \right) T \right\} \right.$$
$$= 0 - -T \, \text{Exp}\left\{ \left(\phi - \Delta - \kappa \right) T \right\}$$
$$= T \, \text{Exp}\left\{ \left(\phi - \Delta - \kappa \right) T \right\} \tag{20}$$

C. The solution to the following equation is...

$$B = \frac{\delta}{\delta\kappa} \left(\kappa + \Delta - \phi \right)^{-1}$$
$$= -\left(\kappa + \Delta - \phi \right)^{-2} \tag{21}$$

D. The solution to the following equation is...

$$C = \frac{\delta}{\delta\kappa} \, \text{Exp}\left\{ -\kappa T \right\}$$
$$= -T \, \text{Exp}\left\{ -\kappa T \right\} \tag{22}$$

References

[1] Gary Schurman, *The Newton Raphson Method For Solving Non-Linear Equations,* October, 2009.

[2] Gary Schurman, *Modeling Debt - Solving For Debt IRR In Discrete-Time,* March, 2023.